Verification of Binarized Deep Neural Networks

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Based on the papers: (1) *Verifying Properties of Binarized Deep Neural Networks* – N. Narodytska et all, AAAI-18, (2) *Formal Analysis of Deep Binarized Neural Networks* – N. Narodytska, IJCAI-18

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Outline

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Motivation

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4. The methods were tested for image classification tasks for medium-size BNNs.

Why Verification/Analyais of Deep Neural Networks (DNNs) is important?

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Verification problem: encode the (exact representation of the) network and the property we aim to verify as a formal statement, using, e.g. ILP, SMT or SAT.

Contents

Notations

- $[m] = \{1, 2, ..., m\}$
- $\mathbf{v} = (v_1, ..., v_m),\ \mathbf{v} \in \mathbb{R}$
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- *Image classification problem*: we are given (1) a set of training images drawn from an unknown distribution ν over $X = \mathbb{Z}^n$, where n is the size of individual images, (2) a label for each image $L : \mathbb{Z}^n \to [s]$.

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  - *Training*: given a labeled training set, learn a neural network classifier that can be used as inference engine.
  - During the inference the network is *fixed*.

Contents

Properties of neural networks

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Properties:

1. Robustness: small perturbations on inputs do not affect the output.
   - global robustness: for any valid input, there is no small perturbation that can change the decision of the network on this input.

Definition (Global Robustness)

A feedforward neural network F is globally-robust if for any input $\mathbf{x}$, $\mathbf{x} \in X$ and τ, $\|\tau\|_\infty \leq \epsilon$ we have that $F(\mathbf{x} + \tau) = l_{\mathbf{x}}$.

   - local robustness: is defined for a single input x.

Definition (Local Robustness)

A feedforward neural network F is locally-robust for an input $\mathbf{x}$, $\mathbf{x} \in X$, if there does not exist τ, $\|\tau\|_\infty \leq \epsilon$ such that $F(\mathbf{x} + \tau) \neq l_x$.

2.

3.

Properties of neural networks

Let F be a general feedforward neural, $F(\mathbf{x})$ be the output of F on input $\mathbf{x}$ and $l_{\mathbf{x}} = L(\mathbf{x})$ be the ground truth label of $\mathbf{x}$.
Properties:

1. Robustness: small perturbations on inputs do not affect the output.
2. Invertibility: explore a set of inputs that map to a given output (example: what the inputs of the network are, if exist, that map to a given output).

Definition (Local Invertibility)

A feedforward neural network F is locally invertible for an output $\mathbf{s}$ if there exists $\mathbf{x}$, $\mathbf{x} \in C(X)$, such that $F(\mathbf{x}) = \mathbf{s}$, where $C(X)$ denotes the constrained domain of inputs.

Related problem: how to enumerate multiple, preferably diverse by some measure, inputs of the network that map to a given output.

3.

Properties of neural networks

Let F be a general feedforward neural, $F(\mathbf{x})$ be the output of F on input $\mathbf{x}$ and $l_{\mathbf{x}} = L(\mathbf{x})$ be the ground truth label of $\mathbf{x}$.

Properties:

1.
2.
3. Network equivalence: two networks F_1 and F_2 are equivalent if they generate same outputs on all inputs drawn from the domain X.

Definition (Network Equivalence)

Two feedforward neural networks F_1 and F_2 are equivalent if for all $\mathbf{x} \in X$, $F_1(\mathbf{x}) = F_2(\mathbf{x})$.

Application: network alteration.

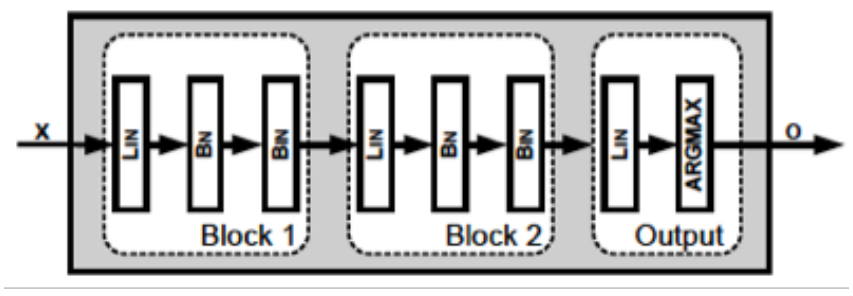
Contents

Binarized Neural Networks (BNNs)

Definition (Binarized Neural Network)

A binarized neural network $BNN : \{-1, 1\}^n \rightarrow [s]$ is a feedforward network that is composed of d blocks, BLK_1, ..., BLK_{d-1}, O. Formally, given an input $\mathbf{x}$, $BNN(\mathbf{x}) = O(BLK_{d_1}, ...(BLK_1(\mathbf{x})))$

Schematic view of a binarized neural network



Structure of internal and outputs blocks, which stacked together form a BNN. A_k and b_k – parameters of the *LIN* layer; $\alpha_{k_i}, \gamma_{k_i}, \mu_{k_i}, \sigma_{k_i}$ – parameters of the *BN* layer. μ and σ correspond to mean and standard deviation computed in the training phase. The *BIN* layer is parameter free.

Structure of kth Internal block, $BLK_k : \{-1,1\}^{n_k} \rightarrow \{-1,1\}^{n_{k+1}}$ on input $x_k \in \{-1,1\}^{n_k}$	
LIN	$y = A_k x_k + b_k$, where $A_k \in \{-1,1\}^{n_{k+1} \times n_k}$ and $b_k \in \mathbb{R}^{n_{k+1}}$
BN	$z_i = \alpha_{k_i}\left(\frac{y_i - \mu_{k_i}}{\sigma_{k_i}}\right) + \gamma_{k_i}$, where $y = (y_1, \ldots, y_{n_{k+1}})$, and $\alpha_{k_i}, \gamma_{k_i}, \mu_{k_i}, \sigma_{k_i} \in \mathbb{R}$. Assume $\sigma_{k_i} > 0$.
BIN	$x_{k+1} = \text{sign}(z)$ where $z = (z_1, \ldots, z_{n_{k+1}}) \in \mathbb{R}^{n_{k+1}}$ and $x_{k+1} \in \{-1,1\}^{n_{k+1}}$

Structure of Output Block, $O : \{-1,1\}^{n_d} \rightarrow [s]$ on input $x_d \in \{-1,1\}^{n_d}$	
LIN	$w = A_d x_d + b_d$, where $A_d \in \{-1,1\}^{s \times n_d}$ and $b_d \in \mathbb{R}^s$
ARGMAX	$o = \text{argmax}(w)$, where $o \in [s]$

Contents

Encodings of BNNs

... into Boolean formulae.

- The encoding of the BNN is a conjunction of encodings of its blocks.
- $BINBLK_k(\mathbf{x}_k, \mathbf{x}_{k+1})$ a Boolean function that encodes the kth block (BLK_k) with an input $\mathbf{x}_k$ and an output $\mathbf{x}_{k+1}$.
- $BINO(\mathbf{x}_d, o)$ be a Boolean function that encodes O that takes an input $\mathbf{x}_d$ and outputs $\mathbf{o}$.
- The entire BNN on input $\mathbf{x}$ can be encoded as a Boolean formula, with $\mathbf{x}_1$ (first layer) $= \mathbf{x}$ (input):

$$\left(\bigwedge_{k=1}^{d-1} BINBLK_k(x_k, x_{k+1}) \right) \wedge BINO(x_d, o)$$

- Encodings: MILP $\rightsquigarrow$ ILP $\rightsquigarrow$ SAT.

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Mixed Integer Linear Program (MILP) Encoding

- *Encoding of BLK_k*: encode each layer in BLK_k to MILP separately. Let a_i be the i-th row of the matrix A_k, $x_k \in \{-1, 1\}^{n_k}$ denote the input to BLK_k.
- *Linear Transformation*. Transformation for fully connected layer (suitable for convolutions also as they are linear operations). We have:

$$y_i = \langle \mathbf{a}_i, \mathbf{x}_k \rangle + b_i, \quad i = 1, ..., n_{k+1} \tag{1}$$

  where $\mathbf{y} = (y_1, ..., y_{n_{k+1}}) \in \mathbb{R}^{n_{k+1}}$.
- *Batch Normalization*: takes the output of the linear layer as an input. By definition, we have:

$$z_i = \alpha_{k_i} \left(\frac{y_i - \mu_{k_i}}{\sigma_{k_i}} \right) + \gamma_{k_i}, \quad i = \overline{1, n_{k+1}}$$

$$\sigma_{k_i} z_i = \alpha_{k_i} y_i - \alpha_{k_i} \mu_{k_i} + \sigma_{k_i} \gamma_{k_i} \tag{2}$$

- *Binarization*. For the BIN operation, which implements a sign function, we need to deal with conditional constraints.

$$z_i \geq 0 \Rightarrow v_i = 1 \tag{3}$$

$$z_i < 0 \Rightarrow v_i = -1 \quad i = \overline{1, n_{k+1}} \tag{4}$$

Mixed Integer Linear Program Encoding (cont'd)

- *Encoding of O*: $w_i = \langle \mathbf{a}_i, \mathbf{x}_d \rangle + b_i$, $i = 1, .., s$, where $\mathbf{a}_i$ represents the i-th column in A_d and $\mathbf{w} = (w_1, ..., w_s)$. To encode ARGMAX, an ordering relation between w_i's must be imposed:

$$w_i \geq w_j \iff d_{ij} = 1 \qquad d_{ij} \text{ newly introduced vars}$$

$$\sum_{j=1}^{s} d_{ij} = s \implies o = i \qquad i, j = \overline{1, s} \tag{5}$$

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Example

Consider an internal block with two inputs and one output. Suppose we have the following parameters: $A_k = [1, -1]$, $b_k = [-0.5]$, $\alpha_k = [0.12]$, $\mu_k = [-0.1]$, $\sigma_k = [2]$, $\delta_k = [0.1]$.

1. apply the linear transformation: $y_1 = x_{k_1} - x_{k_2} - 0.5$
2. apply batch normalization: $2z_1 = 0.12y_1 + (-0.12) * (-0.1) + 2 * 0.1$
3. apply binarization: $z1 \geq 0 \implies v_1 = 1 \wedge z_1 < 0 \implies v_1 = -1$. $(x_{k+1} = v_1)$.

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Example

Consider an output block with two inputs and two outputs. We have the following parameters for this block $A_d = [1, -1; -1, 1]$ and $b = [-0.5, 0.2]$.

1. encoding of the linear transformation

$$w_1 = x_{d_1} - x_{d_2} - 0.5 \qquad w_2 = -x_{d_1} + x_{d_2} + 0.2$$

2. Two outputs $\rightsquigarrow$ 4 Boolean variables d_{ij}, $i, j = 1, 2$; $d_{11} = d_{22} = 1 \rightsquigarrow$ consider only non-diagonal variables.

$$w_1 \geq w_2 \iff d_{12} = 1 \quad \wedge \quad w2 < w1 \iff d_{21} = 1$$

3. we compute the output o of the neural network as:

$$d_{11} + d_{12} = 2 \implies o = 1 \quad \wedge \quad d_{21} + d_{22} = 2 \implies o = 2$$

Contents

Integer Linear Programming (ILP) Encoding

ILP encoding is smaller than the MILP one.

- *Encoding of BLK_k*: z and y are functional variables of $\mathbf{x}_k$. We can substitute them in (3) and (4) based on (1) and (2) respectively:

$$\frac{\alpha_{k_i}}{\sigma_{k_i}}(\langle \mathbf{a}_i, \mathbf{x}_k \rangle + b_i) - \frac{\alpha_{k_i}}{\sigma_{k_i}}\mu_{k_i} + \gamma_{k_i} \Longrightarrow v_i = 1$$

- *Linear and batch normalization:*
  - Case $\alpha_{k_i} > 0$, we have:

    $\langle \mathbf{a}_i, \mathbf{x}_k \rangle \geq -\frac{\sigma_{k_i}}{\alpha_{k_i}}\gamma_{k_i} + \mu_{k_i} - b_i \Longrightarrow x_i^{'} = 1$(see p. 6618 right column, bottom)

    Consider $C_i = \left\lceil -\frac{\sigma_{k_i}}{\alpha_{k_i}}\gamma_{k_i} + \mu_{k_i} - b_i \right\rceil$. We encode (3) and (4):

$$\langle \mathbf{a}_i, \mathbf{x}_k \rangle \geq C_i \Rightarrow v_i = 1$$
$$\langle \mathbf{a}_i, \mathbf{x}_k \rangle < C_i \Rightarrow v_i = -1 \quad i = \overline{1, n_{k+1}}$$

  - Case $\alpha_{k_i} < 0$. Same as above but $C_i = \left\lfloor -\frac{\sigma_{k_i}}{\alpha_{k_i}}\gamma_{k_i} + \mu_{k_i} - b_i \right\rfloor$
  - Case $\alpha_{k_i} = 0$, we have: $\gamma_{k_i} \Longrightarrow v_i = 1$

Integer Linear Programming (ILP) Encoding (cont'd)

- *Encoding of O*: introduce the Boolean variables d_{ij} avoiding the intermediate variables w_i

$$\langle \mathbf{a}_i, \mathbf{x}_d \rangle + b_i \geq \langle \mathbf{a}_j, \mathbf{x}_d \rangle + b_j \iff d_{ij} = 1, \quad i, j = \overline{1, s}$$

$$\Longleftrightarrow$$

$$\langle \mathbf{a}_i - \mathbf{a}_j, \mathbf{x}_d \rangle \geq \lceil b_j - b_i \rceil \iff d_{ij} = 1$$

where $\mathbf{a}_i$ and $\mathbf{a}_j$ denote the ith and jth rows in the matrix A_d.

Further, constraints (5) can be used.

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Example

Recall the internal block with two inputs and one output: $A_k = [1, -1]$, $b_k = [-0.5]$, $\alpha_k = [0.12]$, $\mu_k = [-0.1]$, $\sigma_k = [2]$, $\delta_k = [0.1]$. We have:

$$x_{k_1} - x_{k_2} \geq \left\lceil \frac{-2}{0.1} * 0.1 - 0.1 - (-0.5) \right\rceil = -1 \Rightarrow v_1 = 1$$

$$x_{k_1} - x_{k_2} < \left\lceil \frac{-2}{0.1} * 0.1 - 0.1 - (-0.5) \right\rceil = -1 \Rightarrow v_1 = -1$$

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$$\iff$$

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where $\mathbf{a}_i$ and $\mathbf{a}_j$ denote the ith and jth rows in the matrix A_d.

Further, constraints (5) can be used.

Example

Recall the output block with two inputs and two outputs: $A_d = [1, -1; -1, 1]$ and $b = [-0.5, 0.2]$. We have

$$x_{d_1} - x_{d_2} - 0.5 \geq -x_{d_1} + x_{d_2} + 0.2 \iff d_{12} = 1 \text{ equiv. to } x_{d_1} - x_{d_2} \geq \left\lceil \frac{0.7}{2} \right\rceil \iff d_{12} = 1$$

$$x_{d_2} - x_{d_1} + 0.2 \geq -x_{d_2} + x_{d_1} - 0.5 \iff d_{21} = 1 \text{ equiv. to } x_{d_1} - x_{d_2} \leq \left\lceil \frac{0.7}{2} \right\rceil \iff d_{21} = 1$$

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Sequential counters for encoding cardinality constraints

Consider a *cardinality constraint*: $\sum_{i=1}^{m} l_i \geq C$, where $l_i \in \{0, 1\}$ is a Boolean variable and C is a constant. This can be compiled into CNF using *sequential counters* $SQ(l, C)$, $l = (l_1, ..., l_m)$.

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Then $SQ(l, C)$ is equivalent to:

$$(l_1 \Leftrightarrow r_{(1,1)}) \wedge (\neg r_{(1,j)},\ j = \overline{2, C})$$
$$r_{(i,1)} \Leftrightarrow l_i \vee r_{(i-1,1)} \quad \wedge$$
$$r_{(i,j)} \Leftrightarrow l_i \wedge r_{(i-1,j-1)} \vee r_{(i-1,j)},\ j = \overline{2, C}$$

where $i = \overline{2, m}$, $r_{(j,p)} = \mathbb{T} \iff \sum_{i=1}^{j} l_i \geq p$

SAT Encoding (cont'd)

- *Encoding of BLK_k*:

$$\langle \mathbf{a}_i, \mathbf{x}_k \rangle \geq C_i \iff \sum_{j=1}^{n_k} a_{ij} x_{k_j} \geq C_i \iff$$

Variable replacement: $x_{k_j} \in \{0,1\} \rightsquigarrow x_{k_j}^{(b)} \in \{0,1\}$ with $x_{k_j} = 2x_{k_j}^{(b)} - 1$. We have:

$$\sum_{j=1}^{n_k} a_{ij}(2x_{k_j}^{(b)} - 1) \geq C_i \Rightarrow v_i = 1 \iff$$

Denote: $\mathbf{a}_i^+ = \{j | a_{ij} = 1\}$ and $\mathbf{a}_i^- = \{j | a_{ij} = -1\}$. We have:

$$\sum_{j \in \mathbf{a}_i^+} x_{k_j}^{(b)} - \sum_{j \in \mathbf{a}_i^-} x_{k_j}^{(b)} \geq \left\lceil \frac{C_i}{2} + \sum_{j=1}^{n_k} \frac{a_{ij}}{2} \right\rceil \iff \sum_{j \in \mathbf{a}_i^+} x_{k_j}^{(b)} - \sum_{j \in \mathbf{a}_i^-} (1 - \overline{x_{k_j}^{(b)}}) \geq C_i' \iff$$

$$\sum_{j \in \mathbf{a}_i^+} x_{k_j}^{(b)} - \sum_{j \in \mathbf{a}_i^-} \overline{x_{k_j}^{(b)}} \geq C_i' + |a_i^-|. \text{ Hence } \sum_{j=1}^{n_k} l_{k_j} \geq D_i \Rightarrow v_i^{(b)} = 1, \ i = \overline{1, n_{k+1}}$$

Further we have: $\bigwedge_{i=1}^{n_{k+1}} SQ(I, D_i) \wedge \bigwedge_{i=1}^{n_{k+1}} \left(r_{i(n_k, D_i)} \iff v_i^{(b)} \right)$

Similarly for

$$\langle \mathbf{a}_i, \mathbf{x}_k \rangle < C_i \Rightarrow v_i = -1 \quad i = \overline{1, n_{k+1}}$$

SAT Encoding (cont'd)

Example

Recall the internal block with two inputs and one output: $A_k = [1, -1]$, $b_k = [-0.5]$, $\alpha_k = [0.12]$, $\mu_k = [-0.1]$, $\sigma_k = [2]$, $\delta_k = [0.1]$. We have:

$$x_{k_1} - x_{k_2} \geq -1 \Rightarrow v_1 = 1 \iff 2x_{k_1}^{(b)} - 1 - (2x_{k_2}^{(b)} - 1) \geq -1 \Rightarrow v_1^{(b)} = 1 \iff$$

$$x_{k_1}^{(b)} - x_{k_2}^{(b)} \geq \lceil 0.5 \rceil$$

$$x_{k_1} - x_{k_2} < -1 \Rightarrow v_1 = -1$$

Contents

Speeding-up the SAT Encoding

- Takes advantage of the modular structure of BNNs.

Speeding-up the SAT Encoding

- Takes advantage of the modular structure of BNNs.
- The approach works for all properties, we exemplify for adversarial robustness.

Speeding-up the SAT Encoding

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- The network can be encoded as a conjunction of two Boolean formulas: *Gen* (generator) encodes the first block of the network, and *Ver* (verifier) encodes the rest of the network:

$$BNN_{A_d}(\mathbf{x}+\tau, o, l_{\mathbf{x}}) = Gen(\mathbf{x}+\tau, \mathbf{y}) \wedge Ver(\mathbf{y}, \mathbf{z}, o, l_{\mathbf{x}})$$

where

$$Gen(\mathbf{x}+\tau, \mathbf{y}) = CNF(||\tau||_\infty \leq \epsilon) \wedge \bigwedge_{i=1}^{n} CNF(\mathbf{x}_i + \tau_i) \in [LB, UB]) \wedge$$
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- *Gen* and *Ver* share only $\mathbf{y} \rightsquigarrow$ use *Craig interpolants* to build efficient search procedure.

Speeding-up the SAT Encoding (cont'd)

Definition (Craig Interpolants)

Let A and B be Boolean formulas such that the formula $A \wedge B$ is UNSAT. Then there exists a formula I, called interpolant, such that $vars(I) = vars(A) \cap vars(B)$, $B \wedge I$ is UNSAT and $A \Rightarrow I$. In general, there exist multiple interpolants for the given A and B.

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7. Repeat from step 1. The procedure terminates since the solution space is reduced.
8. If the formula $Gen(\mathbf{x}+\tau, \mathbf{y})$ becomes UNSAT, then there is no valid perturbation τ , i.e., the network is ϵ-robust on image $\mathbf{x}$.

Contents

Encoding the Properties

1. Adversarial Constraint:
   $BNN_{A_d}(\mathbf{x}+\tau, o, l_{\mathbf{x}}) = CNF(||\tau||_\infty \leq \epsilon) \vee \bigwedge_{i=1}^{n} CNF((\mathbf{x}_i+\tau_i) \in [L,U]) \wedge$
   $BNN(\mathbf{x}+\tau, o) \wedge CNF(o \neq l_{\mathbf{x}})$
2. Verifying Universal Adversarial Robustness

$$\bigwedge_{i=1}^{|S|} BNN_{A_d}(\mathbf{x}_i+\tau, o_i, l_{\mathbf{x}_i}) \iff q_j \wedge CNF\left(\sum_{i=1}^{|S|} q_j \geq \rho|S|\right)$$

3. Verifying Network Equivalence: if

$$\bigwedge_{i=1}^{n} CNF(x_i \in [L,U]) \wedge BNN_1(\mathbf{x}, o_1) \wedge BNN_2(\mathbf{x}, o_2) \wedge o_1 \neq o_2$$

   is UNSAT then the networks are equivalent. If SAT then we obtain a witness image $\mathbf{x}$.

Contents

Experimental Results

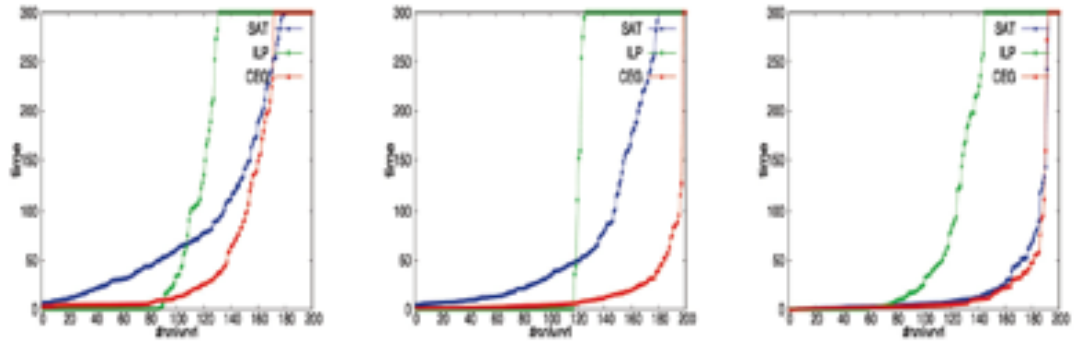
- Torch framework; Tital Pascal X GPU
- Datasets: MNIST, MNIST-rot (MINIST where the digits were rotated uniformly in $[0, 2\pi]$ radians), MNIST-back-image (MINIST with a patch from a black-and-white image was used as the background for the digit image)
- focus on adversarial robustness
- Architecture
  - 4 internal blocks with each block containing a linear layer (LIN) and a final output block.
  - LIN layer in the first block contains 200 neurons, the LIN layers in other blocks contain 100 neurons
  - BN and BIN layers in each block were used; additionally a hard *tanh* layer in each internal block was used only during training
  - For inputs processing, two layers (BN and BIN) were added to the BNN, as the first 2 layers in the network to perform binarization of the grayscale inputs $\rightsquigarrow$ (+) network architecture simplification and search space reduction; (-) lower accuracy of the original BNN by approx.1 %
- Accuracy of the resulting network on the MNIST, MNISTrot, and MNIST-back-image datasets were 95.7%, 71%, resp. 70%

Experimental Results (cont'd)

Checking adversarial robustness:

- from each dataset randomly picked 20 images correctly classified by the network for each of the 10 classes (coresponding to digits) $\rightsquigarrow$ 200 images
- for search space reduction, focus on important pixels as defined by saliency map: perturb the top 50% of highly salient pixels in an image; if a valid perturbation that leads to misclassification among this set of pixels can not be found then search again over all pixels of the image.
- experimented with 3 different maximum perturbation values $\epsilon \in \{1, 3, 5\}$
- timeout: 300 seconds for each instance
- compare three methods of searching for adversarial perturbations. (1) ILP method with SCIP solver, (2) pure SAT method for the sequential counters method using Glucose SAT solver, (3) the SAT menthod from (2) augumented with the counter-example-guided.
- complexity of the SAT formulae: 1.4 million variables and 5 million clauses: MNIST-rot – approx 7 million clauses; MNIST and MNIST-back – approx 5 and 3 million clauses on average, respectively.
- the largest instance contains: 3 million variables and 12 million clauses.

Experimental Results (cont'd)



(a) MNIST, $\epsilon = 1$ (b) MNIST-rot, $\epsilon = 1$ (c) MNIST-back-image, $\epsilon = 1$

	Solved instances (out of 200)									Certifiably ϵ-robust		
	MNIST			MNIST-rot			MNIST-back-image					
	SAT	ILP	CEG	SAT	ILP	CEG	SAT	ILP	CEG	SAT	ILP	CEG
	#solved (t)	#solved (t)	#solved (t)	#solved (t)	#solved (t)	#solved (t)	#solved (t)	#solved (t)	#solved (t)	#	#	#
$\epsilon = 1$	180 (77.3)	130 (31.5)	171 (34.1)	179 (57.4)	125 (10.9)	197 (13.5)	191 (18.3)	143 (40.8)	191 (12.8)	138	96	138
$\epsilon = 3$	187 (77.6)	148 (29.0)	181 (35.1)	193 (61.5)	155 (9.3)	198 (13.7)	107 (43.8)	67 (52.7)	119 (44.6)	20	5	21
$\epsilon = 5$	191 (79.5)	165 (29.1)	188 (36.3)	196(62.7)	170(11.3)	198(13.7)	104 (48.8)	70 (53.8)	116 (47.4)	3	-	4

- Advantages of the method: complete search procedure $\rightsquigarrow$ certify ϵ-robustness $\rightsquigarrow$ there exists no adversarial perturbation technique that can fool the network on these images.
- existing methods: incomplete
- with increasing ϵ, the number of images on which the network is ϵ-robust decreases as the adversary can leverage the larger value to construct